# Short- and long-term relative arbitrage in stochastic portfolio theory

Martin Larsson

Department of Mathematics, ETH Zurich

(with J. Ruf)

D MATH

ETH zürich

Stochastic Portfolio Theory was first introduced by Robert Fernholz. One considers the **market weights**

$$\mu_t^i = \frac{S_t^i}{S_t^1 + \cdots + S_t^d}$$

where $S_t^1, \ldots, S_t^d$ are the market capitalizations of $d$ stocks.

Stochastic Portfolio Theory was first introduced by Robert Fernholz. One considers the **market weights**

$$\mu_t^i = \frac{S_t^i}{S_t^1 + \cdots + S_t^d}$$

where $S_t^1, \ldots, S_t^d$ are the market capitalizations of $d$ stocks.

A basic goal is to find self-financing trading strategies $\theta_t = (\theta_t^1, \ldots, \theta_t^d)$ that perform well relative to the market. The **relative wealth** is

$$V_t^\theta = \theta_t^\top \mu_t = V_0^\theta + \int_0^t \theta_s^\top d\mu_s.$$

There is no bank account, but holding (a constant fraction of) the market portfolio is "relatively risk-free".

Stochastic Portfolio Theory was first introduced by Robert Fernholz. One considers the **market weights**

$$\mu_t^i = \frac{S_t^i}{S_t^1 + \cdots + S_t^d}$$

where $S_t^1, \ldots, S_t^d$ are the market capitalizations of $d$ stocks.

A basic goal is to find self-financing trading strategies $\theta_t = (\theta_t^1, \ldots, \theta_t^d)$ that perform well relative to the market. The **relative wealth** is

$$V_t^\theta = \theta_t^\top \mu_t = V_0^\theta + \int_0^t \theta_s^\top d\mu_s.$$

There is no bank account, but holding (a constant fraction of) the market portfolio is "relatively risk-free".

The market weights are Itô semimartingales $d\mu_t = b_t dt + \sigma_t dW_t$ valued in

$$\overline{\Delta^d} = \{x \in \mathbb{R}_+^d \colon x_1 + \cdots + x_d = 1\}.$$

**Definition.** Given $T \geq 0$, a self-financing trading strategy $\theta$ is a **relative arbitrage over** $[0, T]$ if

$$V_0^\theta = 1,\ V^\theta \geq 0,\ V_T^\theta \geq 1,\ P(V_T^\theta > 1) > 0.$$

**Definition.** Given $T \geq 0$, a self-financing trading strategy $\theta$ is a **relative arbitrage over $[0, T]$** if

$$V_0^\theta = 1,\ V^\theta \geq 0,\ V_T^\theta \geq 1,\ P(V_T^\theta > 1) > 0.$$

**Questions:**

- When does relative arbitrage over $[0, T]$ exist for some $T \geq 0$?
- How small/large can/must $T$ be?
- What does $\theta$ look like? How (if at all) does it depend on the probabilistic properties of $S$ (or $\mu$)?

# Conditions for relative arbitrage over $[0, T]$

**Fernholz '02:** Large enough $T$, provided for some $\delta > 0$, $\varepsilon > 0$,

$$\max_{1 \le i \le d} \mu_t^i \le 1 - \delta, \qquad \lambda_{\min}\left(\frac{d}{dt}\langle \log S \rangle_t\right) \ge \varepsilon \tag{$*$}$$

# Conditions for relative arbitrage over $[0,T]$

**Fernholz '02:** Large enough $T$, provided for some $\delta > 0$, $\varepsilon > 0$,

$$\max_{1\leq i\leq d} \mu_t^i \leq 1-\delta, \qquad \lambda_{\min}\left(\frac{d}{dt}\langle \log S\rangle_t\right) \geq \varepsilon \qquad (*)$$

**Fernholz, Karatzas, Kardaras '05:** Any $T > 0$, still assuming $(*)$.

# Conditions for relative arbitrage over $[0, T]$

**Fernholz '02:** Large enough $T$, provided for some $\delta > 0$, $\varepsilon > 0$,

$$\max_{1 \le i \le d} \mu_t^i \le 1 - \delta, \qquad \lambda_{\min}\left(\frac{d}{dt}\langle \log S \rangle_t\right) \ge \varepsilon \qquad (*)$$

**Fernholz, Karatzas, Kardaras '05:** Any $T > 0$, still assuming $(*)$.

**Fernholz, Karatzas '05:** Large enough $T$, provided for some $\eta > 0$,

$$\sum_{i=1}^{d} \mu_t^i \frac{d}{dt}\langle \log \mu^i \rangle_t \ge \eta \qquad (**)$$

# Conditions for relative arbitrage over $[0, T]$

**Fernholz '02:** Large enough $T$, provided for some $\delta > 0$, $\varepsilon > 0$,

$$\max_{1 \leq i \leq d} \mu_t^i \leq 1 - \delta, \qquad \lambda_{\min}\left(\frac{d}{dt}\langle \log S \rangle_t\right) \geq \varepsilon \tag{$*$}$$

**Fernholz, Karatzas, Kardaras '05:** Any $T > 0$, still assuming $(*)$.

**Fernholz, Karatzas '05:** Large enough $T$, provided for some $\eta > 0$,

$$\sum_{i=1}^{d} \mu_t^i \frac{d}{dt}\langle \log \mu^i \rangle_t \geq \eta \tag{$**$}$$

**Banner, D. Fernholz '08** and **Pal '16**: Short-term relative arbitrage.

# Conditions for relative arbitrage over $[0, T]$

**Fernholz '02:** Large enough $T$, provided for some $\delta > 0$, $\varepsilon > 0$,

$$\max_{1 \le i \le d} \mu_t^i \le 1 - \delta, \qquad \lambda_{\min}\left(\frac{d}{dt}\langle \log S \rangle_t\right) \ge \varepsilon \tag{$*$}$$

**Fernholz, Karatzas, Kardaras '05:** Any $T > 0$, still assuming $(*)$.

**Fernholz, Karatzas '05:** Large enough $T$, provided for some $\eta > 0$,

$$\sum_{i=1}^{d} \mu_t^i \frac{d}{dt} \langle \log \mu^i \rangle_t \ge \eta \tag{$**$}$$

**Banner, D. Fernholz '08** and **Pal '16**: Short-term relative arbitrage.

One might suspect that $(*)$ is an unrealistic condition, while $(**)$ is much better. Is it sufficient for short-term relative arbitrage? Until recently the answer to this question was unknown.

**Theorem (Fernholz, Karatzas, Ruf (FKR) '18).**

The condition $(**)$ is **not** enough to guarantee relative arbitrage over $[0, T]$ for any $T > 0$.

We'll use a condition that is similar to, but not exactly the same as, the condition $(**)$:

The market weight process $d\mu_t = b_t dt + \sigma_t dW_t$ with values in $\overline{\Delta^d}$ is **admissible** if $\mathrm{tr}(\sigma_t \sigma_t^\top) \geq 1$.

We'll use a condition that is similar to, but not exactly the same as, the condition $(**)$:

> The market weight process $d\mu_t = b_t dt + \sigma_t dW_t$ with values in $\overline{\Delta^d}$ is **admissible** if $\mathrm{tr}(\sigma_t \sigma_t^\top) \geq 1$.

We'd like to compute the smallest time horizon beyond which relative arbitrage is always possible:

$$T_* = \inf \left\{ T \geq 0 \colon \begin{array}{l} \text{every admissible market weight process} \\ \text{admits relative arbitrage over } [0, T] \end{array} \right\}$$

We'll use a condition that is similar to, but not exactly the same as, the condition $(**)$:

> The market weight process $d\mu_t = b_t dt + \sigma_t dW_t$ with values in $\overline{\Delta^d}$ is **admissible** if $\mathrm{tr}(\sigma_t \sigma_t^\top) \geq 1$.

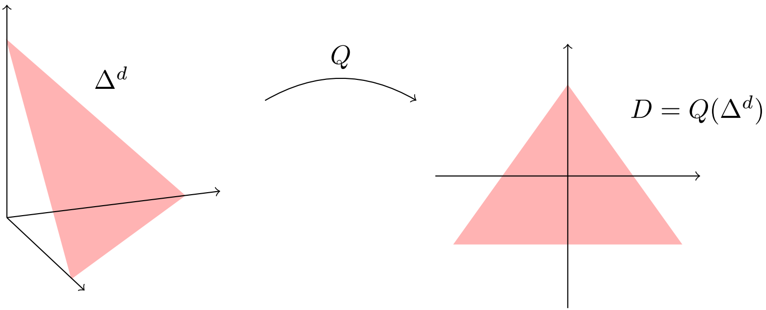
We'd like to compute the smallest time horizon beyond which relative arbitrage is always possible:

$$T_* = \inf \left\{ T \geq 0 \colon \begin{array}{l} \text{every admissible market weight process} \\ \text{admits relative arbitrage over } [0, T] \end{array} \right\}$$

For $d \geq 3$, FKR show that

$$\frac{1}{d(d-1)} \leq T_* \leq 1 - \frac{1}{d}$$

Trading in the market weights $\mu_t^1, \ldots, \mu_t^d$ is equivalent to trading in $1$ "relatively risk-free" asset (the benchmark) and $d-1$ "relatively risky" assets. We make this explicit by a change of coordinates:




$$\mu_t = (\mu_t^1, \ldots, \mu_t^d)$$

$$X_t = (X_t^1, \ldots, X_t^{d-1}) = Q\mu_t$$

## Correspondence between:

| | |
|---|---|
| $d\mu_t = b_t dt + \sigma_t dW_t$ | $dX_t = \beta_t dt + \nu_t dW_t$ |
| self-financing trading in $\mu$ | self-financing trading in $(1, X)$ |
| $V_t^\theta = \theta_t^\top \mu_t = v_0 + \int_0^t \theta_s^\top d\mu_s$ | $V_t^\varphi = v_0 + \int_0^t \varphi_s^\top dX_s$ |
| $\mu$ is admissible, $\mathrm{tr}(\sigma_t \sigma_t^\top) \geq 1$ | $X$ satisfies $\mathrm{tr}(\nu_t \nu_t^\top) \geq 1$ |
| No relative arbitrage exists over $[0, T]$ | $X$ satisfies (NA) on $[0, T]$ |

**Upper bounds** on $T^*$ can be derived using **functionally generated portfolios**.

**Upper bounds** on $T^*$ can be derived using **functionally generated portfolios**.

For $u \in C^2(\mathbb{R}^{d-1})$, Itô's formula states that

$$u(X_0) + \int_0^t \nabla u(X_s)^\top dX_s = u(X_t) - \frac{1}{2}\int_0^t \mathrm{tr}(\nabla^2 u(X_s)\nu_s\nu_s^\top)ds.$$

This is the wealth $V_t^\varphi$ of the self-financing trading strategy $\varphi_t = \nabla u(X_t)$ with initial wealth $u(X_0)$.

**Upper bounds** on $T^*$ can be derived using **functionally generated portfolios**.

For $u \in C^2(\mathbb{R}^{d-1})$, Itô's formula states that

$$u(X_0) + \int_0^t \nabla u(X_s)^\top dX_s = u(X_t) - \frac{1}{2}\int_0^t \mathrm{tr}(\nabla^2 u(X_s)\nu_s \nu_s^\top)ds.$$

This is the wealth $V_t^\varphi$ of the self-financing trading strategy $\varphi_t = \nabla u(X_t)$ with initial wealth $u(X_0)$.

**Example:** Take $u(x) = 1 - \frac{1}{d} - |x|^2 \geq 0$ on $D$. In an admissible model,

$$V_T^\varphi - V_0^\varphi = -u(X_0) + u(X_T) + \int_0^T \mathrm{tr}(\nu_s \nu_s^\top)ds \geq T - (1 - \tfrac{1}{d}).$$

Hence $T_* \leq 1 - \frac{1}{d}$. This is the upper bound of FKR.

**Upper bounds** on $T^*$ can be derived using **functionally generated portfolios**.

For $u \in C^2(\mathbb{R}^{d-1})$, Itô's formula states that

$$u(X_0) + \int_0^t \nabla u(X_s)^\top dX_s = u(X_t) - \frac{1}{2}\int_0^t \mathrm{tr}(\nabla^2 u(X_s)\nu_s\nu_s^\top)ds.$$

This is the wealth $V_t^\varphi$ of the self-financing trading strategy $\varphi_t = \nabla u(X_t)$ with initial wealth $u(X_0)$.

**Example:** Take $u(x) = 1 - \frac{1}{d} - |x|^2 \geq 0$ on $D$. In an admissible model,

$$V_T^\varphi - V_0^\varphi = -u(X_0) + u(X_T) + \int_0^T \mathrm{tr}(\nu_s\nu_s^\top)ds \geq T - (1 - \tfrac{1}{d}).$$

Hence $T_* \leq 1 - \frac{1}{d}$. This is the upper bound of FKR.

**What about lower bounds on $T_*$?**

**Idea:** Let $T > 0$ and suppose $X$ is a martingale on $[0, T]$. This model does not admit relative arbitrage on $[0, T]$, so $T \leq T_*$.

**Idea:** Let $T > 0$ and suppose $X$ is a martingale on $[0, T]$. This model does not admit relative arbitrage on $[0, T]$, so $T \leq T_*$.

**Theorem.** With the notation $\zeta(X) = \inf\{t \geq 0 \colon X_t \notin \overline{D}\}$, one has the representation

$$T_* = \sup \left\{ \operatorname{ess\,inf} \zeta(X) \colon \begin{array}{l} X \text{ is an Itô martingale in } \mathbb{R}^{d-1} \\ \text{with } \frac{d}{dt} \operatorname{tr}\langle X \rangle_t = 1 \end{array} \right\}$$

**Idea:** Let $T > 0$ and suppose $X$ is a martingale on $[0, T]$. This model does not admit relative arbitrage on $[0, T]$, so $T \leq T_*$.

**Theorem.** With the notation $\zeta(X) = \inf\{t \geq 0 \colon X_t \notin \overline{D}\}$, one has the representation

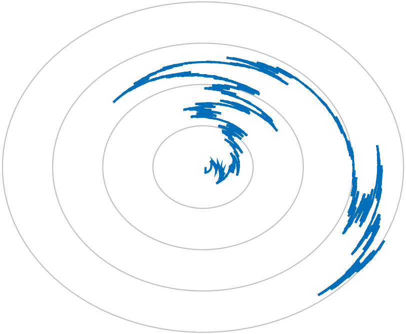
$$T_* = \sup \left\{ \operatorname{ess\,inf} \zeta(X) \colon \begin{array}{l} X \text{ is an Itô martingale in } \mathbb{R}^{d-1} \\ \text{with } \frac{d}{dt} \operatorname{tr}\langle X \rangle_t = 1 \end{array} \right\}$$

But how do we find martingales that don't slow down, yet remain in $\overline{D}$ for a deterministic amount of time?

Here is a 2-dimensional martingale that doesn't slow down, yet stays bounded for deterministic amounts of time:

$$d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \frac{1}{\sqrt{X_t^2 + Y_t^2}} \begin{pmatrix} Y_t \\ -X_t \end{pmatrix} dW_t = \sigma_t dW_t$$

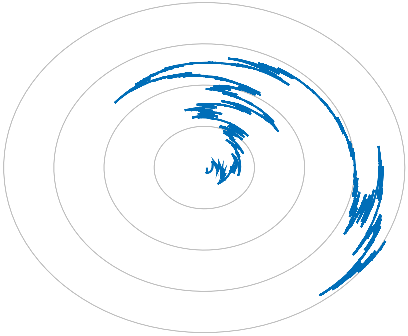
It satisfies $d(X_t^2 + Y_t^2) = \mathrm{tr}(\sigma_t \sigma_t^\top) = |\sigma_t|^2 dt = dt$ and looks like this:



Here is a 2-dimensional martingale that doesn't slow down, yet stays bounded for deterministic amounts of time:

$$d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \frac{1}{\sqrt{X_t^2 + Y_t^2}} \begin{pmatrix} Y_t \\ -X_t \end{pmatrix} dW_t = \sigma_t dW_t$$

It satisfies $d(X_t^2 + Y_t^2) = \operatorname{tr}(\sigma_t \sigma_t^\top) = |\sigma_t|^2 dt = dt$ and looks like this:



... but is poorly adapted to the geometry of $D$.

Focus on $\tau(X) = \inf\{t \geq 0 \colon X_t \notin D\}$ and $d - 1 = 2$.

Focus on $\tau(X) = \inf\{t \geq 0\colon X_t \notin D\}$ and $d - 1 = 2$. Pick $u \in C^2(\mathbb{R}^2)$ and define

$$\nu(x) = \frac{H\nabla u(x)}{|\nabla u(x)|} \qquad \text{where} \qquad H = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Focus on $\tau(X) = \inf\{t \geq 0\colon X_t \notin D\}$ and $d - 1 = 2$. Pick $u \in C^2(\mathbb{R}^2)$ and define

$$\nu(x) = \frac{H\nabla u(x)}{|\nabla u(x)|} \qquad \text{where} \qquad H = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Let $W$ be a 1-dimensional Brownian motion and suppose $X$ solves

$$dX_t = \nu(X_t)dW_t, \qquad X_0 \in D.$$

Focus on $\tau(X) = \inf\{t \geq 0\colon X_t \notin D\}$ and $d-1=2$. Pick $u \in C^2(\mathbb{R}^2)$ and define

$$\nu(x) = \frac{H\nabla u(x)}{|\nabla u(x)|} \qquad \text{where} \qquad H = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Let $W$ be a 1-dimensional Brownian motion and suppose $X$ solves

$$dX_t = \nu(X_t)dW_t, \qquad X_0 \in D.$$

By Itô and since $\nabla u^\top \nu \equiv 0$,

$$t + u(X_t) = u(X_0) + \int_0^t \left(1 + \frac{\nabla u^\top H^\top \nabla^2 u\, H\, \nabla u}{2|\nabla u|^2}(X_s)\right)ds$$

Focus on $\tau(X) = \inf\{t \geq 0 \colon X_t \notin D\}$ and $d - 1 = 2$. Pick $u \in C^2(\mathbb{R}^2)$ and define

$$\nu(x) = \frac{H\nabla u(x)}{|\nabla u(x)|} \qquad \text{where} \qquad H = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Let $W$ be a 1-dimensional Brownian motion and suppose $X$ solves

$$dX_t = \nu(X_t)dW_t, \qquad X_0 \in D.$$

By Itô and since $\nabla u^\top \nu \equiv 0$,

$$t + u(X_t) = u(X_0) + \int_0^t \left(1 + \frac{\nabla u^\top H^\top \nabla^2 u \, H \, \nabla u}{2|\nabla u|^2}(X_s)\right) ds$$

Crucially, **assume** that $(\cdots) = 0$ and $u|_{\partial D} = 0$.

Focus on $\tau(X) = \inf\{t \geq 0 \colon X_t \notin D\}$ and $d - 1 = 2$. Pick $u \in C^2(\mathbb{R}^2)$ and define

$$\nu(x) = \frac{H \nabla u(x)}{|\nabla u(x)|} \qquad \text{where} \qquad H = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Let $W$ be a 1-dimensional Brownian motion and suppose $X$ solves

$$dX_t = \nu(X_t) dW_t, \qquad X_0 \in D.$$

By Itô and since $\nabla u^\top \nu \equiv 0$,

$$t + u(X_t) = u(X_0) + \int_0^t \left(1 + \frac{\nabla u^\top H^\top \nabla^2 u \, H \, \nabla u}{2|\nabla u|^2}(X_s)\right) ds$$

Crucially, **assume** that $(\cdots) = 0$ and $u|_{\partial D} = 0$. Send $t \uparrow \tau(X)$ to get

$$\tau(X) = u(X_0)$$

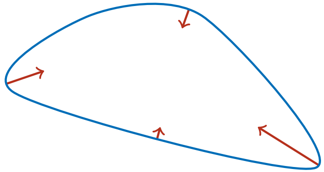
and hence $T_* \geq \sup_{x \in D} u(x)$.

We were hoping to find $u$ such that

$$\begin{cases} 1 + \dfrac{\nabla u^\top H^\top \nabla^2 u \, H \, \nabla u}{2|\nabla u|^2} = 0 & \text{on } D \\ u = 0 & \text{on } \partial D \end{cases}$$

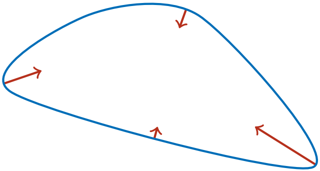
This nonlinear PDE may look complicated. But actually it is equivalent to the so-called arrival time formulation of **mean-curvature flow**.

The mean curvature (or curve shortening) flow deforms an initial contour. Each point on the contour moves in the normal direction at a speed equal to the curvature at that point.

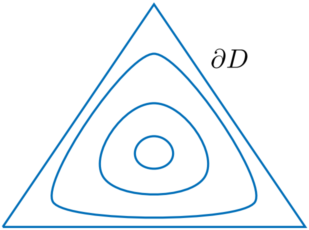


The arrival time $u(x)$ is (twice) the time it takes for the initial contour $\partial D$ to reach the point $x \in D$.

The mean curvature (or curve shortening) flow deforms an initial contour. Each point on the contour moves in the normal direction at a speed equal to the curvature at that point.



The arrival time $u(x)$ is (twice) the time it takes for the initial contour $\partial D$ to reach the point $x \in D$. For our $D$, the contours look like this:



**Theorem.** Let $d = 3$, and let $u \in C^2(D) \cap C(\overline{D})$ be the solution to

$$(*) \qquad \begin{cases} 1 + \dfrac{\nabla u^\top H^\top \nabla^2 u \, H \, \nabla u}{2|\nabla u|^2} = 0 & \text{on } D \\ u = 0 & \text{on } \partial D \end{cases}$$

Then

$$u(x) = \sup \left\{ \operatorname{ess\,inf} \zeta(X) : \begin{array}{l} X \text{ is an Itô martingale in } \mathbb{R}^{d-1} \\ \text{with } \frac{d}{dt} \operatorname{tr}\langle X \rangle_t = 1 \text{ and } X_0 = x \end{array} \right\}$$

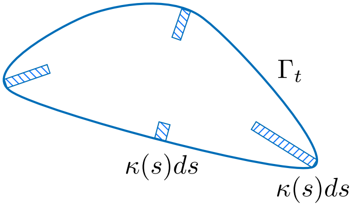
and hence $T_* = \sup_{x \in D} u(x)$.

Mean curvature flow has been studied extensively. Existence, uniqueness, and regularity of the arrival time are well understood. See for instance **Huisken '84**, **Gage & Hamilton '86**, **Evans & Spruck '91**, **Soner & Touzi '03**, **Kohn & Serfaty '06**, **Colding & Minozzi '16, '18**, etc.

Related equations arise as HJB equations in stochastic target problems (**Soner & Touzi '02, '02, '03**) as well as in certain deterministic games (**Kohn & Serfaty '06**).

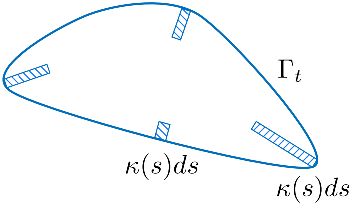
In dimension $d-1=2$ one has the following result, first observed by **Gage & Hamilton '86**. The area $A$ enclosed by smooth simple closed curve that flows by mean curvature satisfies

$$\frac{dA}{dt} = -\int_{\Gamma_t} \kappa(s)ds = -2\pi.$$



In dimension $d-1=2$ one has the following result, first observed by **Gage & Hamilton '86**. The area $A$ enclosed by smooth simple closed curve that flows by mean curvature satisfies

$$\frac{dA}{dt} = -\int_{\Gamma_t} \kappa(s)ds = -2\pi.$$




**Lemma.** The maximal arrival time of the moving front is $A(0)/2\pi$.

**Theorem.** For $d = 3$, the smallest time horizon beyond which any admissible market weight process admits relative arbitrage is

$$T_* = \frac{\sqrt{3}}{2\pi} \approx 0.28.$$

Compare this to the FKR bounds $0.16 \leq T_* \leq 0.67$.

**Theorem.** Fix $d \geq 3$. The value function

$$u(x) = \sup\left\{\operatorname{ess\,inf} \zeta(X) \colon \begin{array}{l} X \text{ is an Itô martingale in } \mathbb{R}^{d-1} \\ \text{with } \operatorname{tr}\langle X\rangle_t \equiv t \text{ and } X_0 = x \end{array}\right\}$$

is the unique outer limiting viscosity solution of the fully nonlinear PDE

$$-1 - \sup\left\{\frac{1}{2}\operatorname{tr}(a\nabla^2 u) \colon a \succeq 0,\ \operatorname{tr}(a) = 1,\ a\nabla u = 0\right\} = 0 \quad \text{in } D$$
$$u = 0 \quad \text{in } \overline{D}^c$$

Hence $T_* = \sup_{x\in D} u(x)$.

This equation describes the "minimum-curvature flow" arrival time. It coincides with the mean-curvature flow equation in the planar case $d = 3$, but not in higher dimension.

For $T > T_*$, relative arbitrage over $[0, T]$ is possible in any admissible model. What do the trading strategies look like?

For $T > T_*$, relative arbitrage over $[0, T]$ is possible in any admissible model. What do the trading strategies look like? Let's try a functionally generated portfolio $\varphi_t = \nabla u(X_t)$ for some $u \in C^2(\mathbb{R}^{d-1})$.

For $T > T_*$, relative arbitrage over $[0, T]$ is possible in any admissible model. What do the trading strategies look like? Let's try a functionally generated portfolio $\varphi_t = \nabla u(X_t)$ for some $u \in C^2(\mathbb{R}^{d-1})$.

$$\begin{aligned} V_t^\varphi &= u(X_0) + \int_0^t \nabla u(X_s)^\top dX_s \\ &= u(X_t) - \int_0^t \frac{1}{2}\operatorname{tr}(\nabla^2 u(X_s)\nu_s\nu_s^\top)ds \\ &\geq u(X_t) - \int_0^t \sup\left\{\frac{1}{2}\operatorname{tr}(\nabla^2 u(X_s)\, a) \colon a \in \mathbb{S}_+^{d-1},\ \operatorname{tr}(a) \geq 1\right\} ds. \end{aligned}$$

For $T > T_*$, relative arbitrage over $[0, T]$ is possible in any admissible model. What do the trading strategies look like? Let's try a functionally generated portfolio $\varphi_t = \nabla u(X_t)$ for some $u \in C^2(\mathbb{R}^{d-1})$.

$$
\begin{aligned}
V_t^\varphi &= u(X_0) + \int_0^t \nabla u(X_s)^\top dX_s \\
&= u(X_t) - \int_0^t \frac{1}{2} \operatorname{tr}(\nabla^2 u(X_s) \nu_s \nu_s^\top) ds \\
&\geq u(X_t) - \int_0^t \sup \left\{ \frac{1}{2} \operatorname{tr}(\nabla^2 u(X_s)\, a) \colon a \in \mathbb{S}_+^{d-1},\ \operatorname{tr}(a) \geq 1 \right\} ds.
\end{aligned}
$$

If $u$ is a nonnegative solution to

$$
-1 - \sup \left\{ \frac{1}{2} \operatorname{tr}(a \nabla^2 u) \colon a \in \mathbb{S}_+^{d-1},\ \operatorname{tr}(a) \geq 1 \right\} = 0 \quad \text{on } D,
$$

we get relative arbitrage over $[0, T]$ for any $T > u(X_0)$. This looks like the equation for mean curvature flow, but ...

The two equations are **not** the same!

$$-1 - \sup\left\{\frac{1}{2}\operatorname{tr}(a\nabla^2 u) \colon a \in \mathbb{S}^2_+,\ \operatorname{tr}(a) \geq 1,\ a\nabla u = 0\right\} = 0$$

$$-1 - \sup\left\{\frac{1}{2}\operatorname{tr}(a\nabla^2 u) \colon a \in \mathbb{S}^2_+,\ \operatorname{tr}(a) \geq 1\right\} = 0$$

The two equations are **not** the same!

$$-1 - \sup\left\{\frac{1}{2}\operatorname{tr}(a\nabla^2 u) \colon a \in \mathbb{S}^2_+,\ \operatorname{tr}(a) \geq 1,\ a\nabla u = 0\right\} = 0$$

$$-1 - \sup\left\{\frac{1}{2}\operatorname{tr}(a\nabla^2 u) \colon a \in \mathbb{S}^2_+,\ \operatorname{tr}(a) \geq 1\right\} = 0$$

The solutions are:

$$u_{\mathrm{ess}}(x) = \sup\left\{\operatorname{ess\,inf} \tau(X) \colon \begin{array}{l} X \text{ is an Itô martingale in } \mathbb{R}^2 \\ \frac{d}{dt}\operatorname{tr}\langle X\rangle_t \geq 1 \text{ and } X_0 = x \end{array}\right\}$$

$$\begin{aligned} u_{\exp}(x) &= \sup\left\{\mathbb{E}[\tau(X)] \colon \begin{array}{l} X \text{ is an Itô martingale in } \mathbb{R}^2 \\ \frac{d}{dt}\operatorname{tr}\langle X\rangle_t \geq 1 \text{ and } X_0 = x \end{array}\right\} \\ &= \frac{2}{3} - |x|^2 \end{aligned}$$

The two equations are **not** the same!

$$-1 - \sup\left\{\frac{1}{2}\operatorname{tr}(a\nabla^2 u)\colon a \in \mathbb{S}^2_+,\ \operatorname{tr}(a) \geq 1,\ a\nabla u = 0\right\} = 0$$

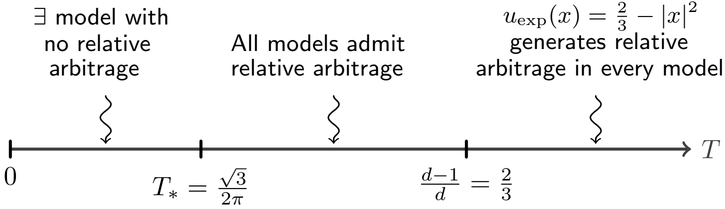
$$-1 - \sup\left\{\frac{1}{2}\operatorname{tr}(a\nabla^2 u)\colon a \in \mathbb{S}^2_+,\ \operatorname{tr}(a) \geq 1\right\} = 0$$

The solutions are:

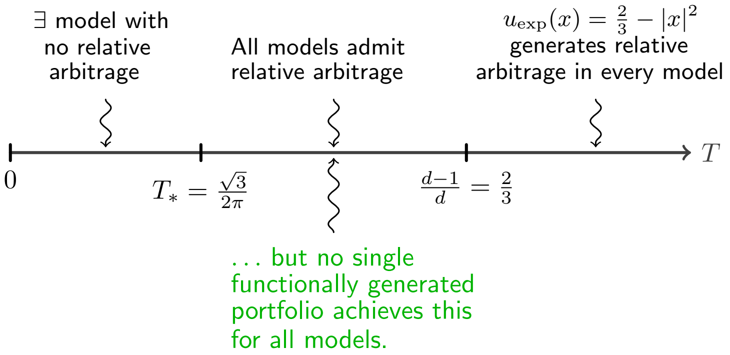
$$u_{\mathrm{ess}}(x) = \sup\left\{\operatorname{ess\,inf} \tau(X)\colon \begin{array}{l} X \text{ is an Itô martingale in } \mathbb{R}^2 \\ \frac{d}{dt}\operatorname{tr}\langle X\rangle_t \geq 1 \text{ and } X_0 = x \end{array}\right\}$$

$$\begin{aligned} u_{\mathrm{exp}}(x) &= \sup\left\{\mathbb{E}[\tau(X)]\colon \begin{array}{l} X \text{ is an Itô martingale in } \mathbb{R}^2 \\ \frac{d}{dt}\operatorname{tr}\langle X\rangle_t \geq 1 \text{ and } X_0 = x \end{array}\right\} \\ &= \frac{2}{3} - |x|^2 \end{aligned}$$

**Conclusion:** The functionally generated portfolio $\varphi_t = \nabla u_{\mathrm{exp}}(X_t)$ only guarantees relative arbitrage over $[0, T]$ for $T > \frac{2}{3} > T_*$. This seems to be optimal among functionally generated portfolios.



∃ model with no relative arbitrage
All models admit relative arbitrage
$u_{\exp}(x) = \frac{2}{3} - |x|^2$ generates relative arbitrage in every model
$0$
$T_* = \frac{\sqrt{3}}{2\pi}$
$\frac{d-1}{d} = \frac{2}{3}$
$T$




**Conjecture:** For every $u \in C^2(D)$, there exists some admissible model $X$ such that $\varphi_t = \nabla u(X_t)$ fails to generate relative arbitrage over $[0, T]$ for all $T < \frac{2}{3}$.

Here is a strategy for proving the conjecture. Fix $u \in C^2(D)$, and look for an admissible model $X$ such that

$$u(X_t) - u(X_0) + \frac{1}{2}\int_0^t \operatorname{tr}(-\nabla^2 u(X_s)\, d\langle X\rangle_s) < 0, \qquad t \in (0, \tfrac{2}{3}).$$

Here is a strategy for proving the conjecture. Fix $u \in C^2(D)$, and look for an admissible model $X$ such that

$$u(X_t) - u(X_0) + \frac{1}{2}\int_0^t \mathrm{tr}(-\nabla^2 u(X_s) d\langle X\rangle_s) < 0, \qquad t \in (0, \tfrac{2}{3}).$$

For this, it's enough to locate a continuous function $\gamma\colon [0, \frac{2}{3}) \to D$ with

$$u(\gamma_t) - u(\gamma_0) + \frac{1}{2}\int_0^t \lambda_{\min}(-\nabla^2 u(\gamma_s)) ds < 0, \qquad t \in (0, \frac{2}{3}).$$

Here is a strategy for proving the conjecture. Fix $u \in C^2(D)$, and look for an admissible model $X$ such that

$$u(X_t) - u(X_0) + \frac{1}{2}\int_0^t \operatorname{tr}(-\nabla^2 u(X_s)d\langle X\rangle_s) < 0, \qquad t \in (0, \tfrac{2}{3}).$$

For this, it's enough to locate a continuous function $\gamma\colon [0, \frac{2}{3}) \to D$ with

$$u(\gamma_t) - u(\gamma_0) + \frac{1}{2}\int_0^t \lambda_{\min}(-\nabla^2 u(\gamma_s))ds < 0, \qquad t \in (0, \frac{2}{3}).$$

We can do this for some functions $u$, including $u_{\exp}(x) = \frac{2}{3} - |x|^2$. Therefore, **for these functions the conjecture is true**.

**Summary:** For admissible models with $d = 3$,

- relative arbitrage always exists beyond $T_* = \frac{\sqrt{3}}{2\pi}$, but not always before $T_*$.
- relative arbitrage is always generated beyond $T = \frac{2}{3}$ using the portfolio generating function $u_{\exp}(x) = \frac{2}{3} - |x|^2$.
- relative arbitrage is possible over $[0, T]$ for $\frac{\sqrt{3}}{2\pi} < T \leq \frac{2}{3}$, but seemingly not by a universal functionally generated portfolio.

**Questions:**

- Form of relative arbitrage strategies in the intermediate regime?
- Other variants of admissibility, like

$$\sum_{i=1}^{d} \mu_t^i \frac{d}{dt} \langle \log \mu^i \rangle_t \geq 1$$

of Fernholz & Karatzas '05, no longer yield mean-curvature flows.

**Thank you!**